String Diagrams for Elementary Category Theory
2: Monads

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Based on joint work with Ralf Hinze

April 3, 2023

Monads

Defined in old school notation

Definition (Monad)

A **monad** on category $\mathcal{C}$ consists of a functor $\mathsf{M} : \mathcal{C} \to \mathcal{C}$ and natural transformations:

$$\eta : \mathsf{Id} \dot{\to} \mathsf{M},$$
$$\mu : \mathsf{M} \circ \mathsf{M} \dot{\to} \mathsf{M}.$$

satisfying **unit** and **associativity** axioms:

$$\mu \cdot (\eta \circ \mathsf{M}) = id = \mu \cdot (\mathsf{M} \circ \eta),$$
$$\mu \cdot (\mu \circ \mathsf{M}) = \mu \cdot (\mathsf{M} \circ \mu).$$

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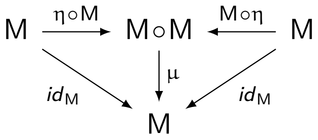

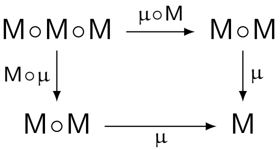

Monads

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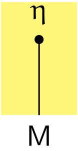
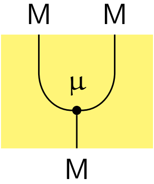


and



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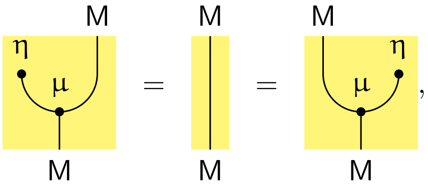
Monads

Defined using string diagrams

Definition (Monad)

satisfying the **unit** equations:

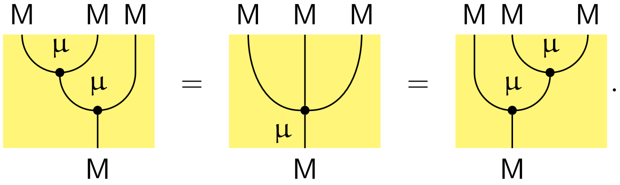


Monads

Defined using string diagrams

Definition (Monad)

and the **associativity** equation:

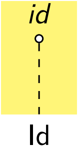


Monads

Examples

Example (The identity monad)

The identity monad on $\mathcal{C}$ consists of the identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ and unit and multiplication both identity natural transformations:



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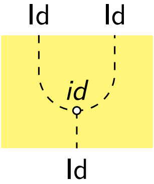

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Example (The list monad)

The functor $\mathsf{L} : \mathbf{Set} \to \mathbf{Set}$ mapping each set X to the set of lists of elements from X is a monad.

- The unit at X, $\eta\, X : X \to \mathsf{L}\,(X)$ maps an element $x \in X$ to the singleton list $[x]$.
- The multiplication at X, $\mu : \mathsf{L}\,(\mathsf{L}\,(X)) \to \mathsf{L}\,(X)$ maps a list of lists to a list by concatenating all its elements. For example:

$$[[1, 2, 3], [4, 5], [6]] \mapsto [1, 2, 3, 4, 5, 6]$$

Example (The multiset monad)

Let the functor $\mathsf{M} : \mathbf{Set} \to \mathbf{Set}$ map each set X to the set of *finite* multisets (or bags) from X. For example:

$$\{ a : 2, b : 1 \}$$

This functor is a monad.

- The unit at X, $\eta\, X : X \to \mathsf{M}\,(X)$ maps an element $x \in X$ to the singleton multiset $\{ x : 1 \}$.
- The multiplication at X, $\mu\, X : \mathsf{M}\,(\mathsf{M}\,(X)) \to \mathsf{M}\,(X)$ maps a multiset of multisets to a multiset by taking "unions" which account for multiplicities. For example:

$$\{ \{ a : 1, b : 2 \} : 2, \{ a : 3, c : 1 \} : 1 \} \mapsto \{ a : 5, b : 4, c : 1 \}$$

Monad morphisms

Definition (Monad morphism)

Given a pair of monads over the *same* base category, $\mathsf{S}, \mathsf{T} : \mathcal{C} \to \mathcal{C}$, a monad morphism is a natural transformation $\tau : \mathsf{S} \dot{\to} \mathsf{T}$ such that:

$$\tau \cdot \eta = \eta \quad \text{and} \quad \tau \cdot \mu = \mu \cdot (\tau \circ \tau)$$

Monad morphisms

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$$
\begin{array}{ccc}
\mathsf{Id}_{\mathcal{C}} & \xrightarrow{\eta} & \mathsf{S} \\
& \searrow^{\eta} & \downarrow{\scriptstyle \tau} \\
& & \mathsf{T}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathsf{S} \circ \mathsf{S} & \xrightarrow{\tau \circ \tau} & \mathsf{T} \circ \mathsf{T} \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
\mathsf{S} & \xrightarrow[\tau]{} & \mathsf{T}
\end{array}
$$

Monad morphisms

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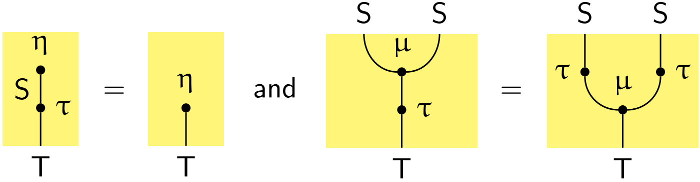
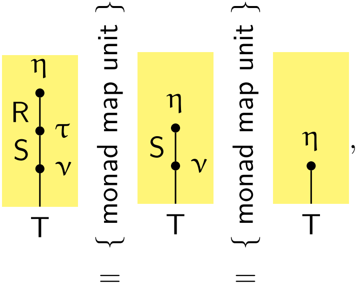


Monad morphisms

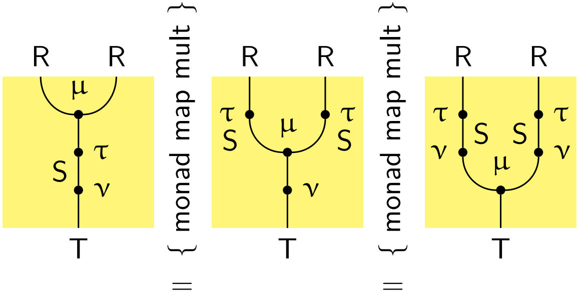
Composition

Unit preservation:



Monad morphisms
Composition

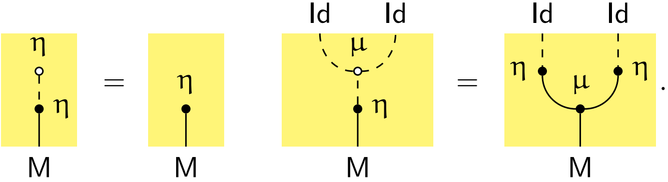
Multiplication preservation:



Monad morphisms

Examples

Example (Monad units)



Monad morphisms

Examples

Example (Lists to multisets)

There is a monad morphism $\mathsf{L} \overset{.}{\to} \mathsf{M}$ from the list to the multiset monad, mapping a list to the multiset of elements that appear, with their multiplicities. For example:

$$[a, b, a, c] \mapsto \{a : 2, b : 1, c : 1\}$$

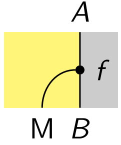
The Kleisli category of a monad

Definition (Kleisli category, part I)

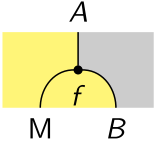
For a monad (M, η, μ), the **Kleisli category**, denoted $\mathcal{C}_{\mathsf{M}}$, has the same objects as $\mathcal{C}$ but the arrows differ. A Kleisli arrow $A \to B : \mathcal{C}_{\mathsf{M}}$ is an arrow of type $A \to \mathsf{M}\, B : \mathcal{C}$ in the underlying category.

The Kleisli category of a monad

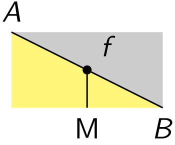
Some options for drawing Kleisli morphisms:



versus



versus



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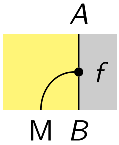
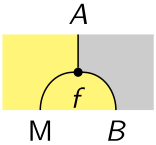
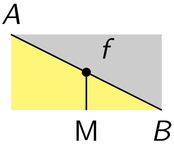


versus



versus



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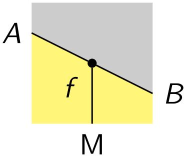
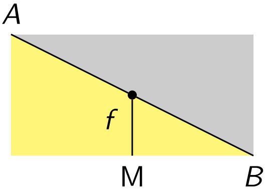
Compact notation:



extends to



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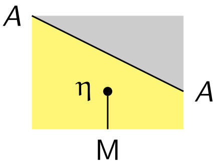
The Kleisli category of a monad

Definition (Kleisli category, part II)

Identity on A in $\mathcal{C}_{\mathsf{M}}$ is given by $\eta\, A : A \to \mathsf{M}\, A : \mathcal{C}$. Graphically:



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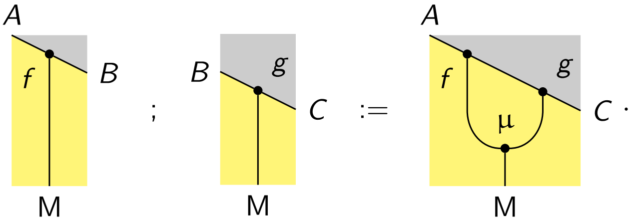
The Kleisli category of a monad

Definition (Kleisli category, part III)

Composition in $\mathcal{C}_{\mathsf{M}}$ is defined as:

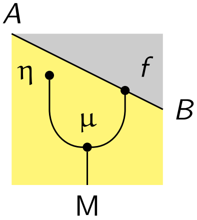


The Kleisli category for a monad

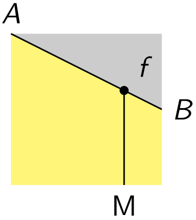
Checking the category axioms

Identity on the left:

$id_A \,;\, f$ = { definition }



= { monad unit }



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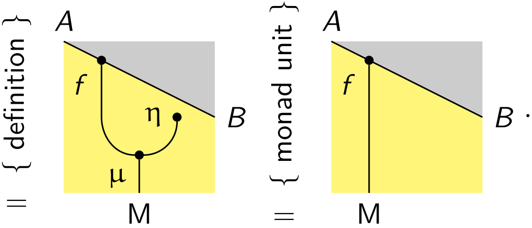
The Kleisli category for a monad

Checking the category axioms

Identity on the right:

$f \, ; id_B$ $=$ { definition }



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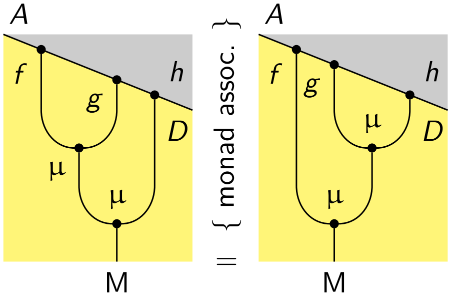
The Kleisli category for a monad

Checking the category axioms

Associativity of composition:

$(f \,;\, g) \,;\, h \;=$



$= \; f \,;\, (g \,;\, h)$

Examples

Example (Identity monad)

The Kleisli category of the identity monad $\mathsf{Id} : \mathbf{Set} \to \mathbf{Set}$ is simply the category of sets and functions.

Examples

Example (List monad)

The Kleisli category of the list monad $\mathsf{L} : \mathbf{Set} \to \mathbf{Set}$ has morphisms the form:

$$X \to LY$$

We can think of this as a nondeterministic computation, mapping an element of X to a list of possible outcomes in Y.

Examples

Example (Multiset monad)

The Kleisli category of the multiset monad $\mathrm{M} : \mathbf{Set} \to \mathbf{Set}$ has morphisms the form:

$$X \to MY$$

We can think of this as a quantified nondeterministic computation, mapping an element of X to a multiset of possible outcomes in Y, tracking the multiplicity of each occurrence.

Algebras for an endofunctor

Definition (F-algebra)

Given an endofunctor $\mathsf{F} : \mathcal{C} \to \mathcal{C}$, an F-**algebra** is a pair (A, a), where A is an object in $\mathcal{C}$, and $a : \mathsf{F}\,A \to A$ is an arrow in $\mathcal{C}$. These are known as the **carrier** and **action** of the algebra respectively.

Algebras for an endofunctor

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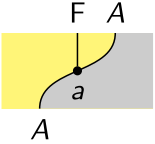

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Algebras for an endofunctor

Definition (F-algebra homomorphism)

An F-**homomorphism** between algebras (A, a) and (B, b) is an arrow $h : A \to B$ in the underlying category $\mathcal{C}$ such that the **homomorphism axiom** holds:

$$h \cdot a = b \cdot \mathsf{F}\, h.$$

Algebras for an endofunctor

Definition (F-algebra homomorphism)

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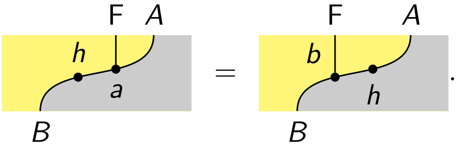
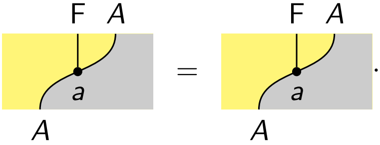
Graphically this condition is:



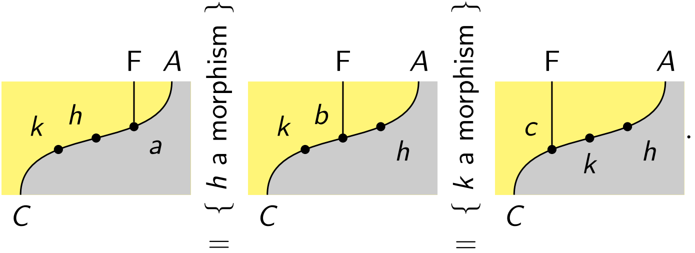
F-algebras form a category

The identity morphisms in $\mathcal{C}$ are F-algebra morphisms. To show to show $a \cdot \mathsf{F}\, id = id \cdot a$ graphically:



F-algebras form a category

F-algebra morphisms compose:



F-algebras form a category

Therefore, for $\mathsf{F} : \mathcal{C} \to \mathcal{C}$, F-algebras and their homomorphisms form a category F-**Alg**($\mathcal{C}$), with composition and identities as in $\mathcal{C}$.

Algebras for a monad

Definition (Eilenberg–Moore algebra)

Given a monad $\mathsf{M} : \mathcal{C} \to \mathcal{C}$, an **algebra for** M, also referred to as an **Eilenberg–Moore algebra for** M, is an M-algebra (A, a) that satisfies the **unit** and **multiplication** axioms, or coherence conditions.

$$a \cdot \eta\, A = id \quad \text{and} \quad a \cdot \mu\, A = a \cdot \mathsf{M}\, a.$$

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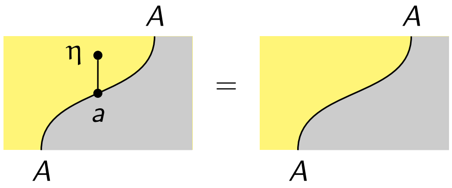
Graphically the unit axiom is:



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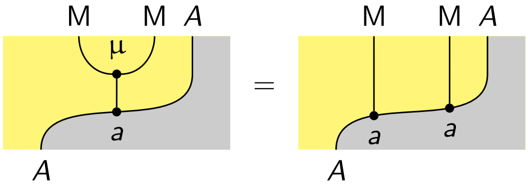
Graphically the multiplication axiom is:



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The category of M-algebras and M-homomorphisms is known as the **Eilenberg–Moore category** of M, denoted $\mathcal{C}^{\mathsf{M}}$.

Algebras for a monad

Example (Identity monad)

The Eilenberg–Moore category of the identity monad $\mathrm{Id} : \mathbf{Set} \to \mathbf{Set}$ is equivalent (in fact isomorphic) to the category $\mathbf{Set}$.

Algebras for a monad

Examples

Example (List monad)

The Eilenberg–Moore category of the list monad $\mathsf{L} : \mathbf{Set} \to \mathbf{Set}$ is equivalent (in fact isomorphic) to the category of monoids and monoid homomorphisms.

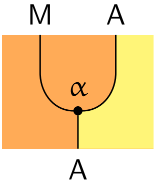
Example (Multiset monad)

The Eilenberg–Moore category of the multiset monad $\mathsf{M} : \mathbf{Set} \to \mathbf{Set}$ is equivalent (in fact isomorphic) to the category of commutative monoids and monoid homomorphisms.

Monad actions

Definition (Left action of a monad)

Given a monad (M, η, μ) on a category $\mathcal{C}$, a **left action of** M on a functor $\mathsf{A} : \mathcal{C} \to \mathcal{D}$ is a natural transformation $\alpha : \mathsf{M} \circ \mathsf{A} \stackrel{.}{\to} \mathsf{A}$:



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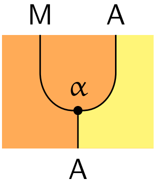

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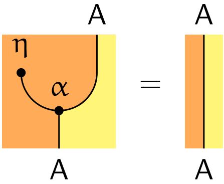
α must respect the unit:



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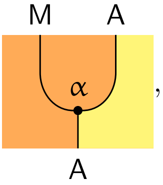
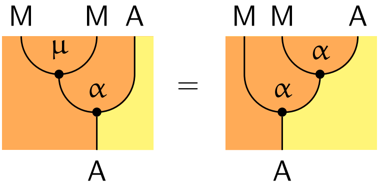
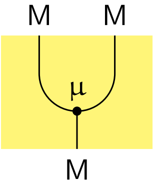


and multiplication:



Example (Multiplication)

For any monad M, the multiplication μ is a (left) action of the monad M on the endofunctor M.

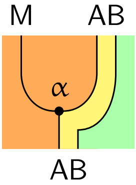


Monad actions

Examples

Example (Outlining on the right)

Given an action $\alpha : \mathsf{M} \circ \mathsf{A} \dot{\to} \mathsf{A}$, we can form new actions by by outlining the right edge as follows.

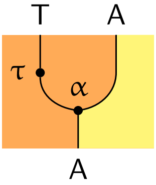


Monad actions

Examples

Example (Dots on the left)

Given an action $\alpha : M \circ A \dot{\to} A$, and a monad (T, η, μ), we can form new actions by placing a dot of type $T \dot{\to} M$ on the left prong. We then consider what axioms τ should satisfy.



Monad actions

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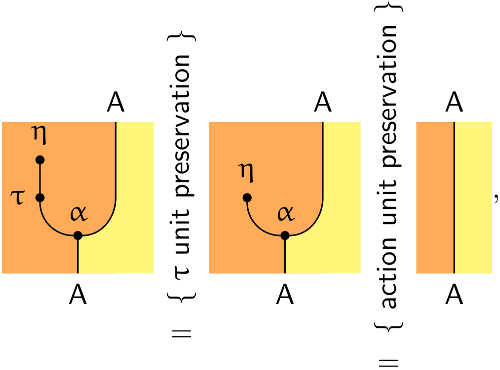
Given an action $\alpha : M \circ A \dot{\to} A$, and a monad (T, η, μ), we can form new actions by placing a dot of type $T \dot{\to} M$ on the left prong. We then consider what axioms τ should satisfy.

Checking the action unit preservation:



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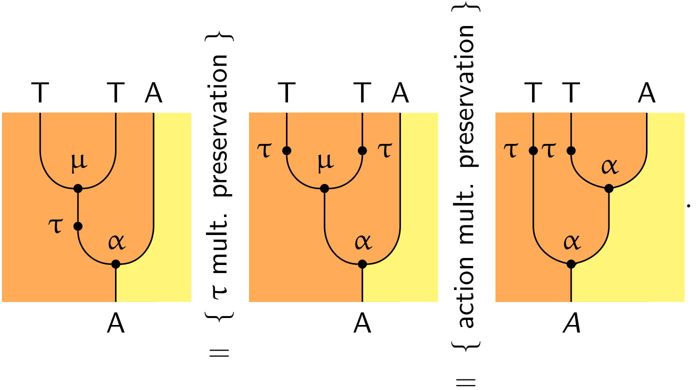
Monad actions

Examples

Example (Dots on the left)

Given an action $\alpha : M \circ A \dot{\to} A$, and a monad (T, η, μ), we can form new actions by placing a dot of type $T \dot{\to} M$ on the left prong. We then consider what axioms τ should satisfy.

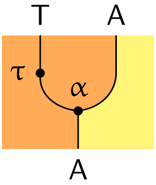
Checking the action multiplication preservation:



Example (Dots on the left)

Given an action $\alpha : M \circ A \overset{\cdot}{\to} A$, and a monad (T, η, μ), we can form new actions by placing a dot of type $T \overset{\cdot}{\to} M$ on the left prong. We then consider what axioms τ should satisfy.

Hence, if τ is a monad morphism, then the composite

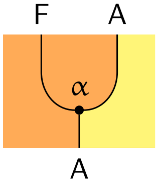


is a valid monad action.

Endofunctor actions

Definition (Vanilla action)

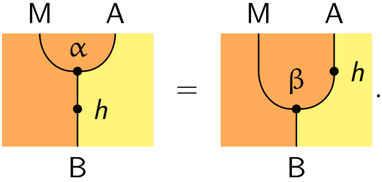
Given an endofunctor F on a category $\mathcal{C}$, a **left action of** F on a functor $A : \mathcal{B} \to \mathcal{C}$ is a natural transformation $\alpha : F \circ A \dot{\to} A$. (Not required to satisfy any additional axioms). In pictures, it is any α of the form:



Transformations of actions

Definition (Transformations of actions)

A **transformation of actions**, written $h : (\mathsf{A}, \alpha) \to (\mathsf{B}, \beta)$, is a natural transformation $h : \mathsf{A} \stackrel{\cdot}{\to} \mathsf{B}$ such that the **right turn axiom** holds:



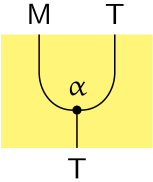
Transformations of vanilla actions are defined analogously.

Relating algebras and actions

F-algebra	vanilla F-action/natural F-algebra
$(A : \mathcal{C}, a : \mathsf{F}\,A \to A)$	$(\mathsf{A} : \mathcal{B} \to \mathcal{C}, \alpha : \mathsf{F} \circ \mathsf{A} \mathrel{\dot{\to}} \mathsf{A})$
algebra for M	left action of M/natural algebra for M
$(A : \mathcal{C}, a : \mathsf{M}\,A \to A)$	$(\mathsf{A} : \mathcal{B} \to \mathcal{C}, \alpha : \mathsf{M} \circ \mathsf{A} \mathrel{\dot{\to}} \mathsf{A})$
homomorphism	transformation of actions/natural homomorphism
$h : (A, a) \to (B, b)$	$h : (\mathsf{A}, \alpha) \to (\mathsf{B}, \beta)$

Compatible actions

Given a monad action:



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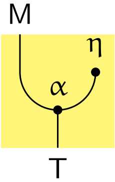
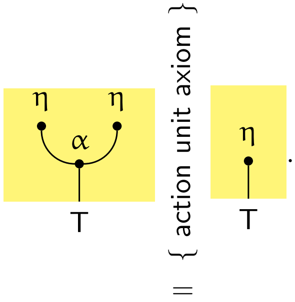
where (T, η, μ) is a monad, when is the composite:



a monad morphism?

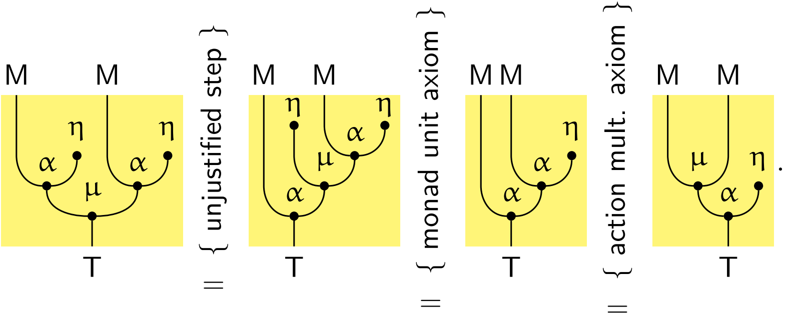
Compatible actions

Checking the monad morphism unit preservation axiom:



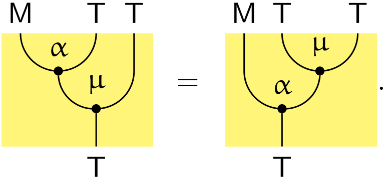
Compatible actions

Attempting the monad morphism multiplication preservation axiom:



Compatible actions

From the previous calculation we have discovered the following **pseudo-associativity** axiom is sufficient:



We shall call such an $\alpha : \mathsf{M} \circ \mathsf{T} \stackrel{.}{\to} \mathsf{T}$ a **compatible action**.

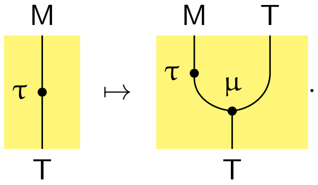
Monad morphisms to compatible actions

Given a monad map $\tau : M \dot{\to} T$, can we construct a compatible action?

Monad morphisms to compatible actions

Given a monad map $\tau : M \dot{\to} T$, can we construct a compatible action?

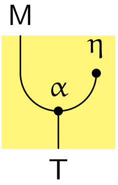
Yes! Via the mapping:



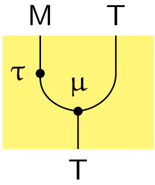
Monad morphisms to compatible actions

In fact, we have a bijection between M-actions compatible with T and monad morphisms $M \stackrel{\cdot}{\to} T$:

$\alpha \mapsto$

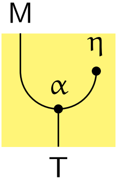


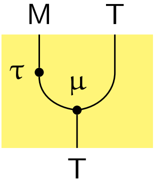
$\tau \mapsto$



Monad morphisms to compatible actions

In fact, we have a bijection between M-actions compatible with T and monad morphisms $\mathsf{M} \mathrel{\dot{\to}} \mathsf{T}$:

$\alpha \mapsto$ 

$\tau \mapsto$ 

In an entirely analogous manner, we can establish a bijection between vanilla actions $\mathsf{F} \circ \mathsf{T} \mathrel{\dot{\to}} \mathsf{T}$ compatible with T, and natural transformations $\mathsf{F} \mathrel{\dot{\to}} \mathsf{T}$.

Next time

Adjunctions!