String Diagrams for Elementary Category Theory

1: Fundamentals

Dan Marsden
Based on joint work with Ralf Hinze

April 2, 2023

Proofs in category theory

Linear syntax

$$
\begin{aligned}
& \mathsf{H}\, h \cdot \mathsf{H}\, a \cdot \lambda\, A \\
= & \quad \{ \text{ functoriality } \} \\
& \mathsf{H}\,(h \cdot a) \cdot \lambda\, A \\
= & \quad \{ \text{ F-morphism } \} \\
& \mathsf{H}\,(b \cdot \mathsf{F}\, h) \cdot \lambda\, A \\
= & \quad \{ \text{ functoriality } \} \\
& \mathsf{H}\, b \cdot (\mathsf{H} \circ \mathsf{F})\, h \cdot \lambda\, A \\
= & \quad \{ \text{ naturality } \} \\
& \mathsf{H}\, b \cdot \lambda\, B \cdot (\mathsf{F} \circ \mathsf{H})\, h
\end{aligned}
$$

Proofs in category theory

Linear syntax

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= \;& \{\ \text{functoriality}\ \} \\
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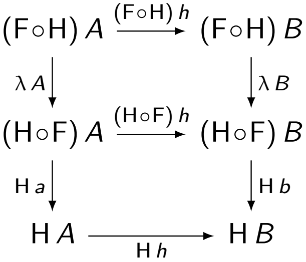
& \mathsf{H}\,b \cdot (\mathsf{H} \circ \mathsf{F})\,h \cdot \lambda\,A \\
= \;& \{\ \text{naturality}\ \} \\
& \mathsf{H}\,b \cdot \lambda\,B \cdot (\mathsf{F} \circ \mathsf{H})\,h)
\end{aligned}
$$

Features

- ✓ Familiar
- ✓ Compact
- ✓ Clear direction of argument
- ✗ Lots of trivial bookkeeping steps
- ✗ Lack of obvious type information
- ✗ Hard to spot possible next steps

Proofs in category theory

Commuting diagrams



Proofs in category theory

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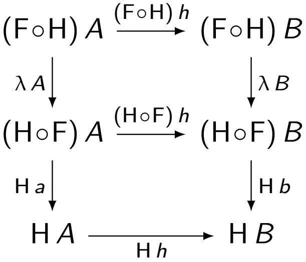
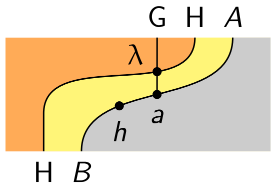


Features

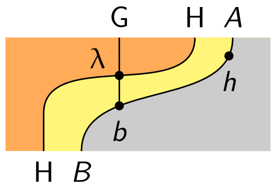
- ✓ Explicit type information
- ✓ Standard in C.T.
- ✗ Less familiar
- ✗ Lack of clear direction
- ✗ Lots of trivial bookkeeping steps
- ✗ Proof steps hard to identity

Proofs in category theory

String diagrams

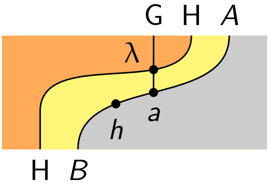


$=$ { F-morphism }



Proofs in category theory

String diagrams



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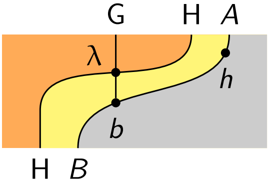

Features

- ✓ Explicit type information
- ✓ Clear direction of proof
- ✓ Less bookkeeping steps
- ✓ Visual clues as to potential proof steps
- ✗ Unfamiliar
- ✗ Might seem less rigorous
- ✗ Initially hard to relate to other notation

Proofs in category theory

Other alternatives

- "Yoneda style" proofs.
- Use of internal languages.
- Higher-level methods and instantiating meta-theory.

This course

Aims

This course is an *applied course*.

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Aims

This course is an *applied course*.

- How to draw and reasoning with string diagrams for elementary category theory.
- *Lots* of examples.
- Various “tricks of the trade”.
- Human friendly.
- No theory of string diagrams.

This course

Lecture contents

1. Basics of string diagrams.

This course

Lecture contents

1. Basics of string diagrams.
2. Monads.

This course

Lecture contents

1. Basics of string diagrams.
2. Monads.
3. Adjunctions.

This course

Lecture contents

1. Basics of string diagrams.
2. Monads.
3. Adjunctions.
4. Some larger examples.

Where do string diagrams come from?

Poincaré dual notational choices

Conventional notation:

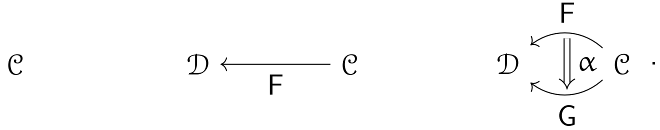
Categories	Functors	Natural Transformations
$\mathcal{C}$	$\mathcal{D} \xleftarrow[F]{} \mathcal{C}$	$\mathcal{D} \underset{G}{\overset{F}{\leftleftarrows}} \mathcal{C}, \quad \alpha : F \Rightarrow G$.

Where do string diagrams come from?

Poincaré dual notational choices

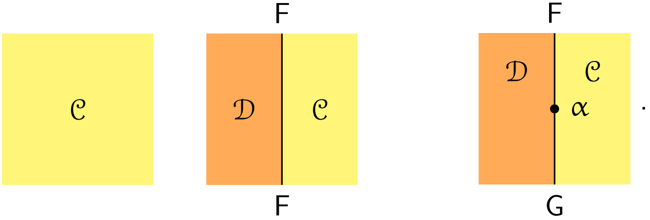
Conventional notation:

Categories | Functors | Natural Transformations



String diagrammatic notation:

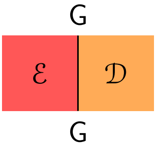
Categories | Functors | Natural Transformations



Functors

Composition

Given functors $\mathsf{G} : \mathcal{D} \to \mathcal{E}$ and $\mathsf{F} : \mathcal{C} \to \mathcal{D}$, in pictures,



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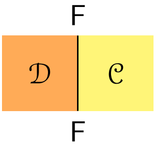
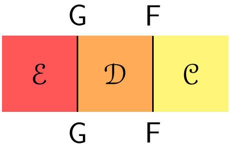


Their composite $\mathsf{G} \circ \mathsf{F} : \mathcal{C} \to \mathcal{E}$ is drawn as follows:



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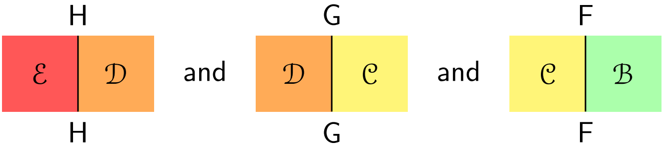
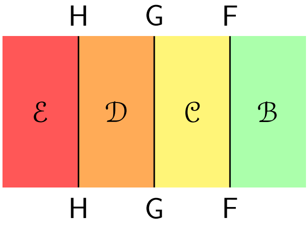
Functors

Associativity of composition

Given three functors $H : \mathcal{D} \to \mathcal{E}$, $G : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{B} \to \mathcal{C}$:



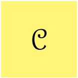
the composites $H \circ (G \circ F)$ and $(H \circ G) \circ F$ are both depicted:



Functors

Identity functors

We draw the identity functor $\mathsf{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ as the corresponding coloured region:



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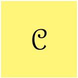
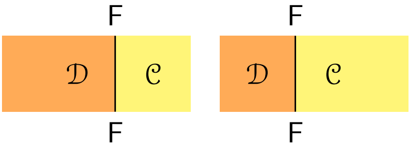


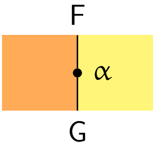
We then can draw $\mathrm{Id}_{\mathcal{D}} \circ F$ and $F \circ \mathrm{Id}_{\mathcal{C}}$ as follows:



Natural transformations

Vertical composition

Given two natural transformations,



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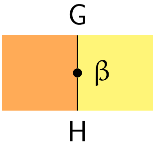

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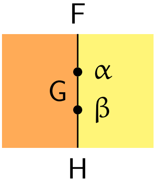
their vertical composite $\beta \cdot \alpha : \mathsf{F} \stackrel{\cdot}{\to} \mathsf{H}$ is drawn:



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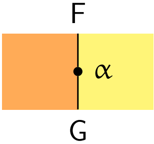
(Recall $(\beta \cdot \alpha)\, A = (\beta\, A) \cdot (\alpha\, A)$).

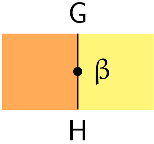
Natural transformations

Associativity of vertical composition

Given three natural transformations,



and



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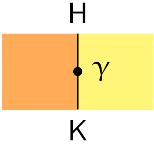

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Natural transformations

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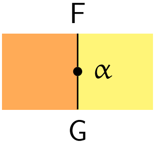

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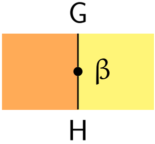

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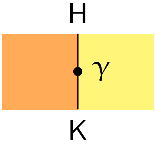

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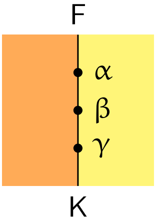
The vertical composites $\gamma \cdot (\beta \cdot \alpha) : F \dot{\to} K$ and $(\gamma \cdot \beta) \cdot \alpha : F \dot{\to} K$ are both drawn:



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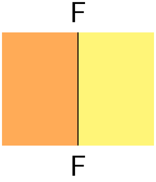
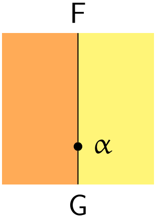
Natural transformations

Vertical composition and identities

We draw the identity natural transformation $id_{\mathsf{F}} : \mathsf{F} \stackrel{\cdot}{\to} \mathsf{F} : \mathcal{C} \to \mathcal{D}$ as the corresponding edge:



The two composites $id_{\mathsf{G}} \cdot \alpha$ and $\alpha \cdot id_{\mathsf{F}}$ are depicted:



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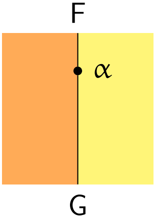
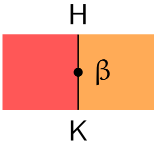
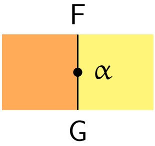


Natural transformations

Horizontal composition

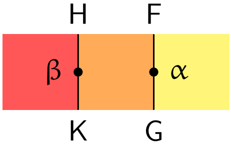


and



,

their horizontal composition $\beta \circ \alpha : \mathsf{H} \circ \mathsf{F} \mathrel{\dot{\to}} \mathsf{K} \circ \mathsf{G}$ is drawn:



.

(Recall $(\beta \circ \alpha)\, A = (\beta\, (\mathsf{G}\, A)) \cdot (\mathsf{H}\, (\alpha\, A)) = (\mathsf{K}\, (\alpha\, A)) \cdot (\beta\, (\mathsf{F}\, A))$.)

Natural transformations

Bifunctoriality of composition

Vertical and horizontal composition satisfy the interchange law

$$(\delta \cdot \gamma)\circ(\beta \cdot \alpha) = (\delta\circ\beta) \cdot (\gamma\circ\alpha).$$

Natural transformations

Bifunctoriality of composition

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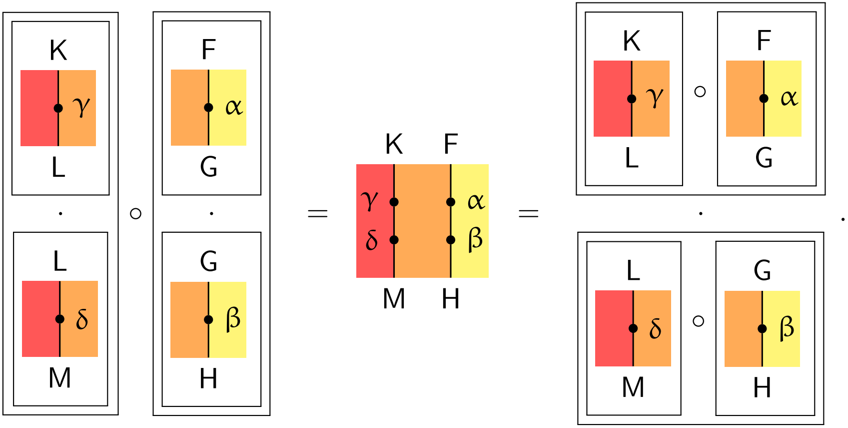
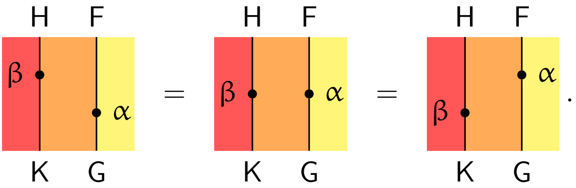
Graphically:



Natural transformations

Elevator Equations

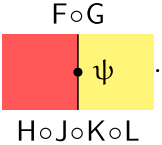
As a special case of bifunctoriality, we have:



Natural transformations

Composite domains and codomains

We could depict natural transformation $\psi : F \circ G \xrightarrow{\cdot} H \circ J \circ K \circ L$ as:



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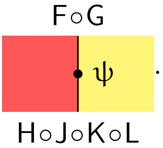
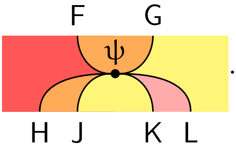


Instead, we should split the wires:



Objects and arrows

Special functors and natural transformations

- The category $\mathbf{1}$ is the category with one object $\star$ and only an identity arrow.
- A functor $X : \mathbf{1} \to \mathcal{C}$ simply specifies a $\mathcal{C}$-object $X\star$.
- A natural transformation between two such functors $f : X \stackrel{.}{\to} Y$ specifies a single $\mathcal{C}$-morphism $f_\star : X\star \to Y\star$. The naturality condition trivially requires:

$$\begin{array}{ccc} X\star & \xrightarrow{id} & X\star \\ {\scriptstyle f_\star}\downarrow & & \downarrow{\scriptstyle f_\star} \\ Y\star & \xrightarrow[id]{} & Y\star \end{array}$$

Objects and arrows

A simple idea

We will reserve the following region to depict the category $\mathbf{1}$:

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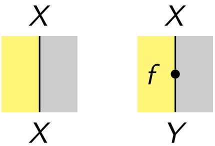
Using our previous observations, we can draw objects and arrows in category $\mathcal{C}$ as:



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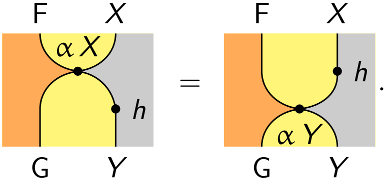
Transformations

Establishing naturality

A transformation $\alpha\, X : \mathsf{F}\, X \to \mathsf{G}\, X$ is natural if for all $h : X \to Y$ the following equation holds:



Symbols to diagrams

Example (A moderately complex term)

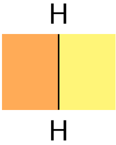
Given functors $\mathsf{H} : \mathcal{C} \to \mathcal{D}$, $\mathsf{S} : \mathcal{C} \to \mathcal{C}$, $\mathsf{T} : \mathcal{D} \to \mathcal{D}$ and natural transformations $\mu : \mathsf{S} \circ \mathsf{S} \stackrel{.}{\to} \mathsf{S}$, and $\delta : \mathsf{T} \circ \mathsf{H} \stackrel{.}{\to} \mathsf{H} \circ \mathsf{S}$, we would like to draw the string diagram corresponding to:

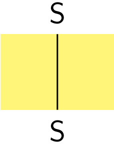
$$(\mathsf{H} \circ \mu) \cdot (\delta \circ \mathsf{S}) \cdot (\mathsf{T} \circ \delta),$$

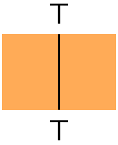
Symbols to diagrams

Example (Step one - the basic building blocks)

The functors:

$\mathsf{H} =$ 

$\mathsf{S} =$ 

$\mathsf{T} =$ 

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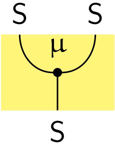
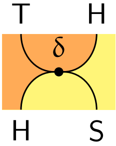
The natural transformations:

$\mu : \mathsf{S} \circ \mathsf{S} \overset{\cdot}{\to} \mathsf{S} =$ 

$\delta : \mathsf{T} \circ \mathsf{H} \overset{\cdot}{\to} \mathsf{H} \circ \mathsf{S} =$ 

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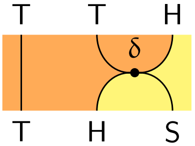
Symbols to diagrams

Example (Step two - the horizontal composites)

$\mathsf{T} \circ \delta \;=$



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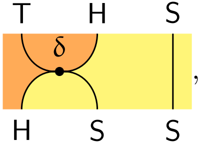
$\delta \circ \mathsf{S} \;=$



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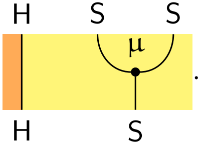
$\mathsf{H} \circ \mu \;=$



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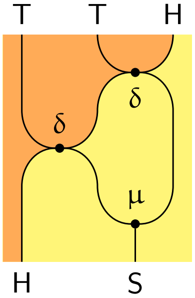
Symbols to diagrams

Example (Step three - the vertical composites)

The diagram for composite

$$(H \circ \mu) \cdot (\delta \circ S) \cdot (T \circ \delta)$$

is:



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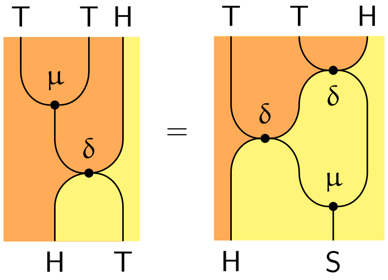
Artistic choices matter

A simple but important idea

How a diagram is drawn can *greatly* aid understanding:

Drawing helpful diagrams

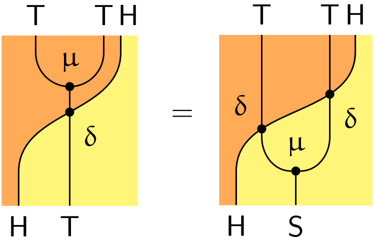


Artistic choices matter

A simple but important idea

How a diagram is drawn can *greatly* aid understanding:

Drawing helpful diagrams



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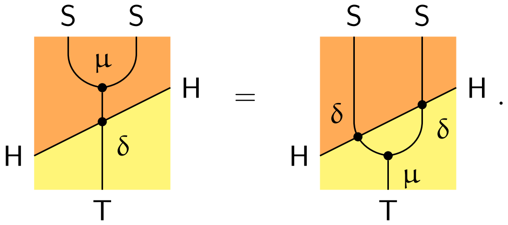
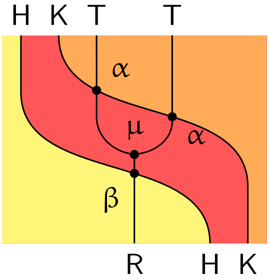


Diagrams to symbols

Example (Converting a moderately complex diagram)



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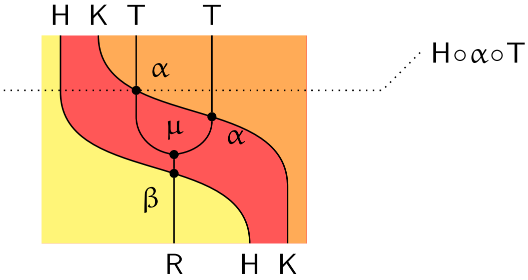

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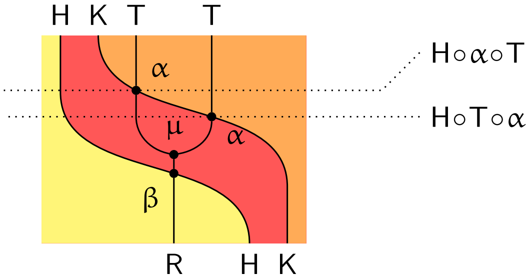

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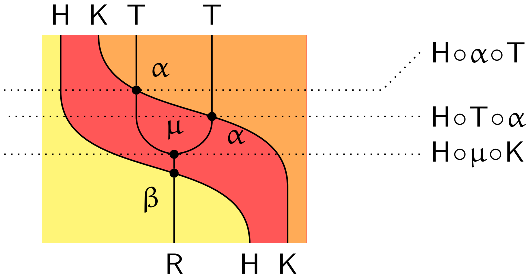

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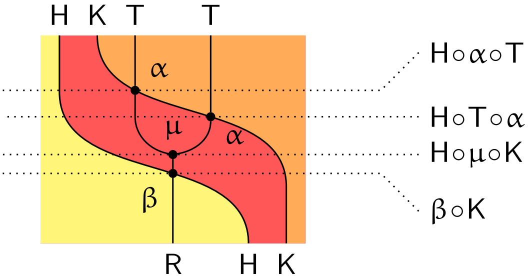

Diagrams to symbols

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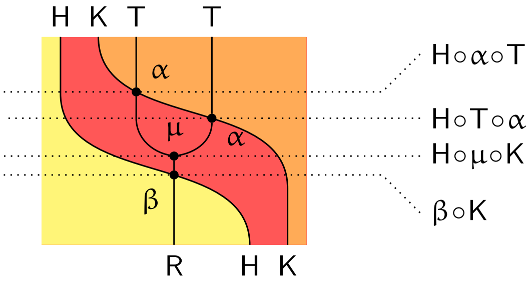
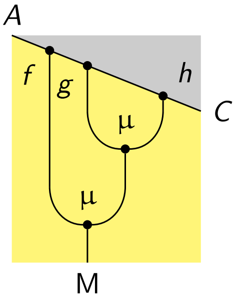


Composing these vertically, we recover the term

$$(\beta \circ K) \cdot (H \circ \mu \circ K) \cdot (H \circ T \circ \alpha) \cdot (H \circ \alpha \circ T).$$

Diagrams to symbols

Example (A more liberal diagram)



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Diagrams to symbols

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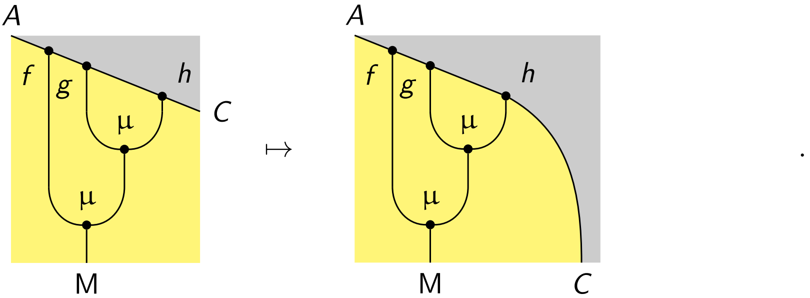
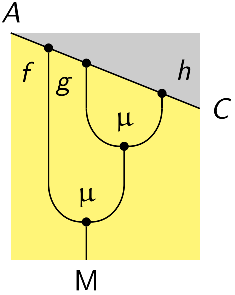
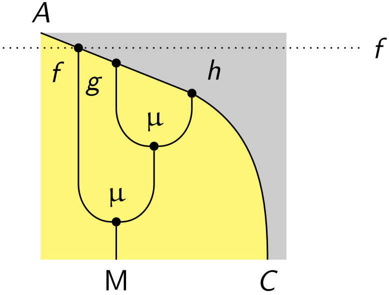


Diagrams to symbols

Example (A more liberal diagram)



$$\mapsto$$



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Diagrams to symbols

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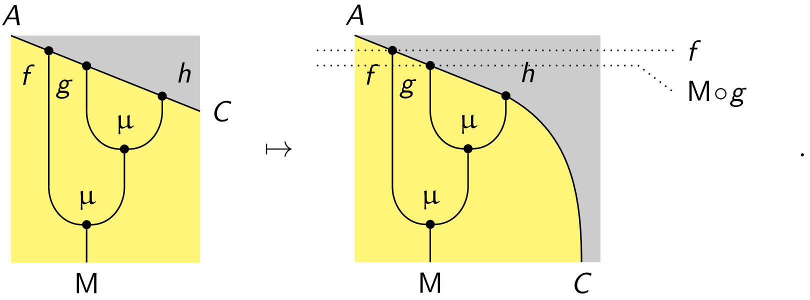

Diagrams to symbols

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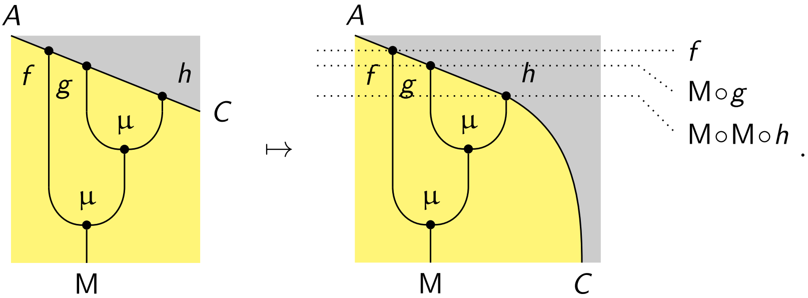

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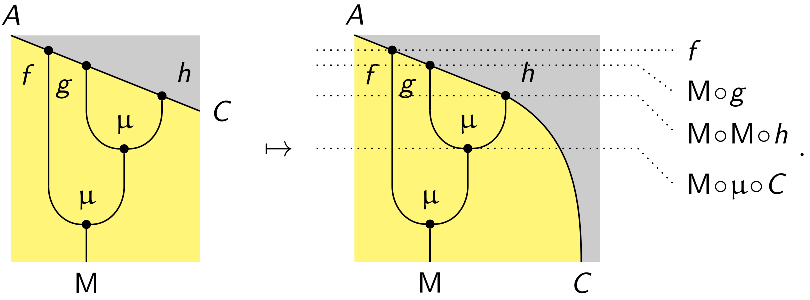
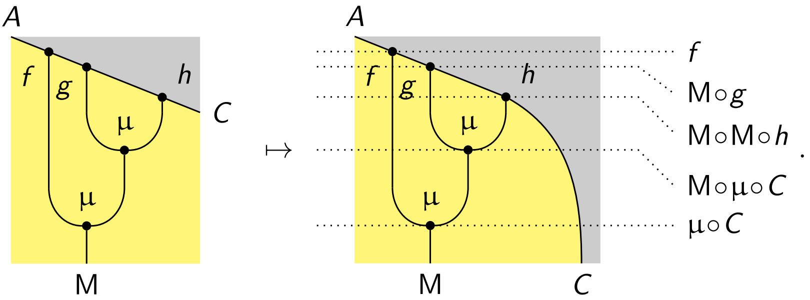


Diagrams to symbols

Example (A more liberal diagram)



This results in symbolic term

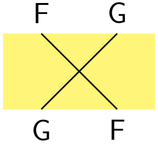
$$(\mu \circ C) \cdot (\mathsf{M} \circ \mu \circ C) \cdot (\mathsf{M} \circ \mathsf{M} \circ h) \cdot (\mathsf{M} \circ g) \cdot f.$$

Things you can't do

The Spengler principle

Don't cross the streams (Dr. E. Spengler)

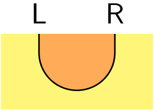
You are not *automatically* allowed to cross wires:



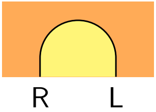
Things you can't do

Don't bend things too far

You are not *automatically* allowed to bend wires back in the direction they came from:



and



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Next time

Monads!