String Diagrams - Exercise Sheet 4

Exercise 1

For functors $I : \mathcal{C} \to \mathcal{D}$ and $\mathsf{G} : \mathcal{C} \to \mathcal{E}$, the right Kan extension of G along I consists of a functor written $\mathsf{G}/I : \mathcal{D} \to \mathcal{E}$ and a natural transformation

$$run : (\mathsf{G}/I) \circ I \stackrel{.}{\to} \mathsf{G}.$$

These two things have to satisfy the following *universal property*: for each functor $\mathsf{F} : \mathcal{D} \to \mathcal{E}$ and for each natural transformation $\alpha : \mathsf{F} \circ I \stackrel{.}{\to} \mathsf{G}$ there exists a natural transformation $[\alpha] : \mathsf{F} \stackrel{.}{\to} \mathsf{G}/I$ (pronounced "shift α") such that

$$\alpha = run \cdot \beta \circ I \iff [\alpha] = \beta, \tag{1}$$

for all $\beta : \mathsf{F} \stackrel{.}{\to} \mathsf{G}/I$.

(In general G/I may not exist. In the questions below, it is assumed the required Kan extensions exist.)

Part I

Introduce suitable notation, and write equation (1) in string diagrams.

Part II

Prove the following three properties of G/I:

1. The **computation law**:
$$\alpha = run \cdot [\alpha] \circ I. \tag{2}$$
2. The **reflection law**:
$$[run] = id. \tag{3}$$
3. The **fusion law**:
$$[\alpha] \cdot \gamma = [\alpha \cdot \gamma \circ I], \tag{4}$$

   Do these properties suggest any changes to your notation?

Part III

Prove the universal property (1) is equivalent to the computation (2), reflection (3) and fusion (4) laws.

Part IV

Show that for any functor $\mathsf{J} : \mathcal{C} \to \mathcal{D}$, J/J carries the structure of a monad on $\mathcal{D}$.

Part V

Assume $\mathsf{L} \dashv \mathsf{R}$. Show:

- $\mathsf{Id}/\mathsf{R} = \mathsf{L}$
- $\mathsf{R}/\mathsf{R} = \mathsf{R} \circ \mathsf{L}$
- That using the construction in part IV yields the same monad as that given by Huber's construction.