String Diagrams - Exercise Sheet 3

Exercise 1

For adjunctions $\mathsf{L} \dashv \mathsf{R} : \mathcal{C} \rightharpoonup \mathcal{D}$ and $\mathsf{L}' \dashv \mathsf{R}' : \mathcal{C}' \rightharpoonup \mathcal{D}'$, and functors $\mathsf{F} : \mathcal{D} \to \mathcal{D}'$ and $\mathsf{G} : \mathcal{C} \to \mathcal{C}'$, show that there are bijections between the sets of natural transformations of type:

1. $\mathsf{L}' \circ \mathsf{F} \stackrel{.}{\to} \mathsf{G} \circ \mathsf{L}$.
2. $\mathsf{F} \stackrel{.}{\to} \mathsf{R}' \circ \mathsf{G} \circ \mathsf{L}$.
3. $\mathsf{L}' \circ \mathsf{F} \circ \mathsf{R} \stackrel{.}{\to} \mathsf{G}$.
4. $\mathsf{F} \circ \mathsf{R} \stackrel{.}{\to} \mathsf{R}' \circ \mathsf{G}$.

Exercise 2

Show that if $\mathsf{L} \dashv \mathsf{R} : \mathcal{C} \rightharpoonup \mathcal{D}$ then L carries the structure of a *right* M-action, where $\mathsf{M} = \mathsf{R} \circ \mathsf{L}$ is the monad structure given by Huber's construction.

Exercise 3

Part I

Show that if $\mathsf{R} : \mathcal{C} \to \mathcal{D}$ is a monad, and $\mathsf{L} \dashv \mathsf{R}$, then L carries the structure of a comonad.

Part II

Show that, for the same conditions as the previous part, L carries the structure of a *right* R-action.

Exercise 4

Two categories $\mathcal{C}$ and $\mathcal{D}$ are **equivalent** , written $\mathcal{C} \simeq \mathcal{D}$, if there is a pair of functors $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ and $\mathsf{G} : \mathcal{C} \leftarrow \mathcal{D}$ and a pair of natural isomorphisms $\alpha : \mathsf{F} \circ \mathsf{G} \cong \mathsf{Id}_{\mathcal{D}}$ and $\beta : \mathsf{G} \circ \mathsf{F} \cong \mathsf{Id}_{\mathcal{C}}$.

Part I

Draw the string diagrams showing α and β are isomorphisms.

Part II

Categories $\mathcal{C}$ and $\mathcal{D}$ are said to be **adjoint equivalent**, written $\mathsf{F} \dashv \mathsf{G} : \mathcal{C} \simeq \mathcal{D}$ if functors F and G form an adjunction $\mathsf{F} \dashv \mathsf{G} : \mathcal{C} \rightharpoonup \mathcal{D}$ where the unit and counit are both isomorphisms.

Show that every pair of functors witnessing an equivalence also witness and adjoint equivalence.