String Diagrams - Exercise Sheet 2

Exercise 1

Part I

A **comonad** consists of an endofunctor $\mathsf{N} : \mathcal{C} \to \mathcal{C}$ and natural transformations:

$$\epsilon : \mathsf{N} \dot{\to} \mathsf{Id},$$
$$\delta : \mathsf{N} \dot{\to} \mathsf{N} \circ \mathsf{N}.$$

such that the following equations hold:

$$(\mathsf{N} \circ \epsilon) \cdot \delta = id = (\epsilon \circ \mathsf{N}) \cdot \delta, \tag{1a}$$
$$(\delta \circ \mathsf{N}) \cdot \delta = (\mathsf{N} \circ \delta) \cdot \delta. \tag{1b}$$

- Drawing the corresponding string diagrams for Equations (1a) and (1b).
- Identity a suitable notion of comonad morphism, and draw the corresponding string diagrams for any equations which must hold.

Exercise 2

Part I

Let $\mathsf{S} : \mathcal{C} \to \mathcal{C}$ and $\mathsf{T} : \mathcal{C} \to \mathcal{C}$ be monads, and $\sigma : \mathsf{S} \to \mathsf{T}$ a monad morphism. Show that σ induces:

- A functor of type $\mathcal{C}_{\mathsf{S}} \to \mathcal{C}_{\mathsf{T}}$.
- A functor of type $\mathcal{C}^{\mathsf{T}} \to \mathcal{C}^{\mathsf{S}}$.

Part II

Let $\mathsf{S} : \mathcal{C} \to \mathcal{C}$ and $\mathsf{T} : \mathcal{D} \to \mathcal{D}$ be monads, and $\mathsf{H} : \mathcal{C} \to \mathcal{D}$ a functor between their base categories. Generalizing the previous part, find sufficient conditions such that:

- H lifts to a functor $\mathcal{C}_{\mathsf{S}} \to \mathcal{D}_{\mathsf{T}}$.
- H lifts to a functor $\mathcal{C}^{\mathsf{S}} \to \mathcal{D}^{\mathsf{T}}$.

Exercise 3

Part I

Let $\mathsf{S} : \mathcal{C} \to \mathcal{C}$ and $\mathsf{T} : \mathcal{C} \to \mathcal{C}$ be monads. Attempt to form a composite monad on $\mathsf{T} \circ \mathsf{S}$ by drawing the "obvious" string diagrams for the unit and multiplication of the composite monad. What goes wrong?

Part II

Find sufficient conditions to address the issue identified in the first part, under which $\mathsf{T} \circ \mathsf{S}$ does carry the structure of a monad.